

Calabi-Yau threefolds in positive characteristic

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Abstract. In this note, an overview of Calabi-Yau varieties in positive characteristic is presented. Although Calabi-Yau varieties in characteristic zero are unobstructed, there are examples of Calabi-Yau threefolds in positive characteristic which cannot be lifted to characteristic zero, although one-dimensional and two-dimensional Calabi-Yau varieties, i.e., elliptic curves and K3 surfaces, are all liftable to characteristic zero. In this respect, Calabi-Yau threefolds in positive characteristic are interesting in view of deformation theory and they are still very mysterious.

Key words: Calabi-Yau variety, positive characteristic, projective lifting problem

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1. Introduction

Calabi-Yau varieties, in particular Calabi-Yau 3-folds, over complex numbers have been extensively studied over the last decades mostly from the interest in mathematical physics. The most notable property of complex Calabi-Yau varieties is that they are unobstructed in deformation.

However, for Calabi-Yau varieties of dimension ≥ 3 in positive characteristic, the situation is quite different. In late 1990's, M. Hirokado found an example of Calabi-Yau 3-fold in characteristic 3 that cannot be lifted to characteristic 0 [16]. After that other examples of non-liftable Calabi-Yau 3-folds have been found [17, 37, 18, 19]. They are all in characteristic $p = 2$ and $p = 3$. However, D. van Straten, S. Cynk and M. Schütt [5, 6] found a large number of examples of non-liftable Calabi-Yau algebraic spaces in characteristic $p \geq 5$, which are not schemes anymore. It is still an open problem whether there exist non-liftable Calabi-Yau varieties in characteristic ≥ 5 .

On the other hand, by the celebrated theorem by P. Deligne and L. Illusie (and M. Raynaud) [8], for a projective variety X over an algebraically closed field k of $\text{char}(k) = p \geq \dim X$, if X can be lifted to the ring $W_2(k)$ of second Witt vectors, Hodge-to-de Rham spectral sequence degenerates at E_1 and Kodaira-Akizuki-Nakano vanishing of cohomologies holds. We note that W_2 -liftable is not a necessary condition for Kodaira type vanishing and there are examples of varieties that are W_2 -non-liftable but Kodaira vanishing holds.

Thus, for non-liftable Calabi-Yau varieties, even if they are not liftable over the ring $W(k)$ of Witt vectors, which implies liftability to characteristic 0, it is an interesting question whether they are liftable over $W_2(k)$. The above mentioned Hirokado's example and the Schröer's examples are known to be non-liftable over $W_2(k)$ [10] and W_2 -liftable is still open for other examples. So Kodaira type vanishing for non-liftable Calabi-Yau varieties in positive characteristic is still a mysterious problem.

In this note, we give an overview of the research on Calabi-Yau 3-folds in positive characteristic in the last decades. The configuration of this note is as follows. In section 2, we summarize briefly what is different from the geometry of Calabi-Yau varieties in characteristic 0. Among the unique features of geometry in positive characteristic, we will focus on obstructedness of deformation, which we will elaborate in section 3. Final section is devoted to construction of non-liftable Calabi-Yau 3-folds. We first summarize the known reasons for non-liftability and overview the examples that have been found so far. We also mention what is known about Kodaira type vanishing for non-liftable Calabi-Yau varieties.

2. Unique features of CY 3-folds in positive characteristic

In this section, we overview unique features of Calabi-Yau 3-folds in positive characteristic as compared to characteristic 0 case. Some of the feature will be considered in detail in the subsequent sections.

Definition 1 (Calabi-Yau n -fold). A *Calabi-Yau n -fold* X is a smooth projective variety over an algebraically closed field k such that $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n = \dim X$ and the canonical

sheaf is trivial $\omega_X \cong \mathcal{O}_X$. In particular, a Calabi-Yau 1-fold is an elliptic curve and a Calabi-Yau 2-fold is a K3 surface.

In the following, we will mostly consider the case of $n = 3$ and $\text{char}(k) = p > 0$. Some authors assume only properness instead of projectivity for Calabi-Yau varieties. But in this note we will always assume projectivity for a Calabi-Yau variety.

2.1. Hodge diamond without Hodge symmetry

By Serre duality, we know that the Hodge numbers $h^{ij} := \dim_k H^k(X, \Omega_X^i)$ of a Calabi-Yau 3-fold X are as follows:

$$\begin{array}{ccccccccc}
 & & & & h^{00} & & & & \\
 & & & & h^{10} & & h^{01} & & \\
 & & & h^{20} & h^{11} & & h^{02} & & \\
 h^{30} & & h^{21} & & h^{12} & & h^{03} & & \\
 & h^{31} & & h^{22} & & h^{13} & & & \\
 & & h^{32} & & h^{23} & & & & \\
 & & & h^{33} & & & & &
 \end{array}
 =
 \begin{array}{ccccccccc}
 & & & & & & & & 1 \\
 & & & & & & h^{10} & & 0 \\
 & & h^{20} & & h^{11} & & & & 0 \\
 1 & & h^{12} & & h^{12} & & & & 1 \\
 & 0 & & h^{11} & & h^{20} & & & \\
 & & 0 & & h^{10} & & & & \\
 & & & & & & & & 1
 \end{array}$$

In positive characteristic, Hodge symmetry ($h^{ij} = h^{ji}$) does not hold in general and we have

Proposition 2 (Hodge symmetry). *For a Calabi-Yau 3-fold X , Hodge symmetry holds if and only if $h^{10} = h^{20} = 0$, namely $H^0(X, \Omega_X^1) = H^0(X, T_X) = 0$, where $T_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ is the tangent bundle.*

Proof. The last part uses the isomorphism $\Omega_X^2 \cong \Omega_X^3 \wedge \Omega_X^{-1} = \mathcal{O}_X \wedge \Omega_X^{-1} \cong T_X$. \square

2.2. obstructed deformation

2.2.1. characteristic 0 case

We first recall the notion of universal deformation.

Definition 3 (deformation). Let X be a complex manifold.

1. A *deformation* of X is a smooth proper morphism $\psi : \mathcal{X} \rightarrow (S, 0)$, where \mathcal{X} and S are connected complex spaces and $0 \in S$ a distinguished point, such that $X \cong \mathcal{X}_0 := \psi^{-1}(0)$,
2. A deformation $\mathcal{X} \rightarrow (S, 0)$ is called *universal* if any other deformation $\psi' : \mathcal{X}' \rightarrow (S', 0')$ is isomorphic to the pull-back under a uniquely determined morphism $\varphi : S' \rightarrow S$ with $\varphi(0') = 0$. We denote a universal deformation by $\mathcal{X} \rightarrow \text{Def}(X)$

Theorem 4 (Bogomolov-Tian-Todorov). *Let X be a Calabi-Yau manifold of any dimension over algebraically closed field k of $\text{char}(k) = 0$. Then $\text{Def}(X)$ is a germ of a smooth manifold with tangent space $H^1(X, T_X)$.*

Proof. By Lefschetz principle and GAGA, we may consider X as a compact complex Kähler manifold. Then we prove smoothness of $\text{Def}(X)$ by a complex analytic method. For the detail, see Theorem 14.10 [11] and the references cited there. \square

According to deformation theory, obstruction is contained in $H^2(X, T_X)$. So if $H^2(X, T_X) = 0$ we can say that X is unobstructed. But Theorem 4 claims that even if $H^2(X, T_X) \neq 0$, which is actually possible in dimension ≥ 3 , its elements are not an obstruction to deformation.

Remark 5. In the algebraic setting, Theorem 4 means that any deformation of X is unobstructed in the sense that for any small extension $\varphi : B \rightarrow A$, namely, for any surjective homomorphism φ of local Artinian \mathbb{C} -algebras with $\mathfrak{m}_B \text{Ker } \varphi = 0$, any variety X_A over $\text{Spec}(A)$ can be lifted over $\text{Spec } B$.

2.2.2. Elliptic curves

For curves, we have $H^2(X, T_X) = 0$ for the dimensional reason so that smooth projective curves in any characteristic are unobstructed. In particular elliptic curves are unobstructed.

2.2.3. K3 surfaces

For a K3 surface X , we have $H^2(X, T_X) = 0$ by the following theorem, so that it is unobstructed.

Theorem 6 (Deligne [7]). *Let X be a K3 surface over an algebraically closed field k of $\text{char}(k) = p > 0$. Then*

1. Hodge to de Rham spectral sequence

$$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H_{DR}^{p+q}(X/k)$$

degenerates at E_1 . In particular, we have the Hodge decomposition

$$H_{DR}^k(X/k) \cong \sum_{i=0}^k H^{k-i}(X, \Omega_X^i)$$

2. the Hodge diamond is

$$\begin{array}{ccccc} & & h^{00} & & \\ & h^{10} & & h^{01} & \\ h^{20} & & h^{11} & & h^{02} \\ & 0 & & h^{12} & \\ & & h^{22} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Unobstructedness $H^2(X, T_X) = 0$ in Theorem 6 implies that a K3 surface over k can be formally lifted over the ring $W(k)$ of Witt vectors, i.e., there exists a formal scheme $\hat{\mathcal{X}}$ over $W(k)$ such that $\hat{\mathcal{X}} \times_{W(k)} k \cong X$. Moreover, $\dim_k H^1(X, T_X) = 20$ means that the moduli space of formal K3 surfaces is 20 dimensional.

On the other hand, if we want an *algebraic* lifting of a K3 surface, the situation is a little subtler.

Projective lifting problem: Let X be a projective variety over a field k of positive characteristic. Then, find a projective scheme \mathcal{X} over $S = \operatorname{Spec} R$, where R is a ring of mixed characteristic, such that $X \cong \mathcal{X} \times_S k$.

Recall that a ring R of mixed characteristic means an integral domain in $\operatorname{char}(R) = 0$ with a maximal ideal $\mathfrak{m} \subset R$ satisfying $\operatorname{char}(R/\mathfrak{m}) > 0$. Here, we can consider $R = W(k)$, for example. In this case, we also need to lift an ample line bundle on X to the formal lifting $\hat{\mathcal{X}}$ and apply Grothendieck's algebraization theorem (see Théorème (5.4.5) [12] or Theorem 21.2 [14]) to obtain a projective scheme \tilde{X} whose formal completion at the closed fiber X is $\hat{\mathcal{X}}$.

The obstruction to lifting an invertible sheaf is contained in $H^2(X, \mathcal{O}_X)$ (see Theorem 24 below) which is 1-dimensional for a K3 surface. Thus, the moduli space of *algebraic* K3 surfaces is smaller by one dimension, namely 19 dimensional.

Projective lifting problem for K3 surfaces has been solved except the case $p = 2$.

Theorem 7 (Ogus [31]). *For $p > 2$, a K3 surface X can be projectively lifted over $W(k)$.*

Proof. By Corollary 2.3 [31], a K3 surface X can be lifted over $W(k)$ if X is not “superspecial”. By Remark 2.3 [31], if Tate conjecture for smooth proper surfaces [45] holds, the only “superspecial” K3 surface is the Kummer surface associated to a product of supersingular elliptic curves and we can show that this can be lifted over $W(k)$. Finally, Tate's conjecture has been established for $p \geq 3$ by several authors. \square

See, for example [24], for the detail of deformation theory of K3 surfaces in positive characteristic.

2.2.4. CY n -folds ($n \geq 3$)

Apart from the cases of $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) = p > 0$ with $\dim \leq 2$, which we have seen so far, the situation is quite different for $\dim \geq 3$. Namely,

Theorem 8 (Hirokado, Schröer, Ito, Saito, Ekedahl, Cynk, van Straten, Schütt). *There are examples of Calabi-Yau 3-folds over an algebraically closed field k of characteristic $p = 2, 3$, that cannot be (formally) lifted to characteristic 0.*

Question: Are there non-liftable Calabi-Yau 3-fold in $p \geq 5$?

By the time when the author writes this article, we only know non-liftable 3-dimensional Calabi-Yau spaces, (i.e., algebraic spaces which are not schemes) in the case $p \geq 5$. Moreover, whether there exist non-liftable CY n -folds for $n \geq 4$ is unclear.

2.3. Hodge decomposition and Kodaira-Akizuki-Nakano vanishing

Let

$$F : X \longrightarrow X^{(p)}$$

be a relative Frobenius morphism. For a complex L^\bullet and $n \in \mathbb{Z}$ we denote by $T^\bullet := \tau_{<n} L^\bullet$ the complex such that

$$T^i = \begin{cases} L^i & \text{for } i \leq n-2 \\ \text{Ker}(d : L^{n-1} \longrightarrow L^n) & \text{for } i = n-1 \\ 0 & \text{for } i \geq n \end{cases}$$

Also, we denote by W_2 the ring of second Witt vectors $W(k)/p^2W(k)$, where $W(k)$ is the ring of Witt vectors over k .

Theorem 9 (Deligne-Illusie [8]). *Let k be a perfect field of $\text{char}(k) = p > 0$ and X a smooth scheme over k . If X is liftable over W_2 , then we have*

$$\varphi : \bigoplus_{i < p} \Omega_{X^{(p)}/k}^i[-i] \xrightarrow{\cong} \tau_{<p} F_* \Omega_{X/S}^\bullet$$

which is an isomorphism in the derived category $D(X^{(p)})$ of $\mathcal{O}_{X^{(p)}}$ -modules with F action such that

$$\mathcal{H}^i \varphi = C^{-1} : \bigoplus_i \Omega_{X^{(p)}/k}^i \longrightarrow \bigoplus_i \mathcal{H}^i F_* \Omega_{X/k}^\bullet$$

for $i < p$, where C is the Cartier operator.

From this theorem, we obtain the following consequences.

Corollary 10 (Hodge decomposition). *Let X be a smooth proper scheme over a perfect field k of $\text{char}(k) = p > 0$ and $\dim \leq p$. If X is liftable over W_2 , then Hodge to de Rham spectral sequence*

$$E_1^{pq} = H^q(X, \Omega^p) \Rightarrow H_{DR}^{p+q}(X/k)$$

degenerates at E_1 . In particular, we have Hodge decomposition:

$$H_{DR}^n(X/k) \cong \bigoplus_{i=0}^n H^{n-i}(X, \Omega^i)$$

for $n \in \mathbb{Z}$.

Corollary 11 (Kodaira-Akizuki-Nakano vanishing). *Let X be a smooth projective scheme over a perfect field k of $\text{char}(k) = p > 0$ and L an ample line bundle on X . If X is liftable over W_2 , then we have*

$$H^j(X, \Omega^i \otimes L^{-1}) = 0 \quad \text{for } i + j < \inf(\dim X, p).$$

In particular, if $\dim X \leq p$, we have Kodaira vanishing $H^i(X, L^{-1}) = 0$ for $i < \dim X$.

For Calabi-Yau 3-folds, the following question is widely open, apart from partial answers.

Question: For a non-liftable Calabi-Yau 3-fold X , is it liftable over W_2 ? If not, does Kodaira vanishing hold?

2.4. Supersingularity

For a Calabi-Yau n -fold X ($n \geq 2$), Artin-Mazur functor [2]

$$\Phi_X^n : \text{Art} \longrightarrow \text{Abgr}$$

from the category of Artinian local rings Art to the category of abelian groups Abgr is defined by

$$\Phi_X^n(S) := \text{Ker}(H_{et}^n(X \times_k S, \mathbb{G}_m) \longrightarrow H_{et}^n(X, \mathbb{G}_m))$$

and this is pro-representable by a 1-dimensional formal group scheme M . Namely, we have

$$\Phi_X^n(-) = \text{Hom}(-, M).$$

It is known that a 1-dimensional formal group scheme M is the formal additive group scheme $\hat{\mathbb{G}}_a$ or a p -divisible formal group scheme. The group operation of a formal group scheme can be described by a formal group law $F(X, Y) \in k[[X, Y]]$ and we can define the *height* of the formal group law. By definition, *height* $h(X)$ of a Calabi-Yau variety X is the height of the formal group law. See [15] for the detail of formal group law for the group scheme M .

The height $ht(X)$ has convenient description in terms of Serre cohomologies $H^*(X, W\mathcal{O}_X)$ [38] and crystalline cohomologies $H_{cris}^*(X/W)$ [3, 4], from which we deduce characterization of supersingular Calabi-Yau varieties.

We will denote by K the quotient field of $W := W(k)$.

Proposition 12. *For a Calabi-Yau variety X of dimension $n = \dim X$ over an algebraically closed field k , we have*

$$ht(X) = \begin{cases} \dim_K H^n(X, W\mathcal{O}_X) \otimes_W K \\ \quad = \dim_K (H_{cris}^n(X/W) \otimes_W K)_{[0,1)} (< \infty) & \text{if } H^n(X, W\mathcal{O}_X) \otimes_W K \neq 0 \\ \infty & \text{if } H^n(X, W\mathcal{O}_X) \otimes_W K = 0 \end{cases}$$

where $(-)_{[0,1)}$ denotes the K -vector subspace of slopes between 0 and 1.

Definition 13 (ordinary/supersingular). Let X be a Calabi-Yau variety of $\dim X = n (\geq 2)$. Then, we call X is *ordinary* if $h(X) < \infty$ and *supersingular* if $h(X) = \infty$.

Proposition 14 (cf.[1]). *If $n = 2$, i.e., X is a K3 surface, we have $1 \leq h(X) \leq 10$ or $h(X) = \infty$. The supersingular case $h(X) = \infty$ exists only in positive characteristic.*

On only K3 surfaces but also for higher dimensional Calabi-Yau varieties, supersingularity has close relation with unique feature of geometry in positive characteristic.

For a Calabi-Yau variety X in characteristic 0, the Betti number $b_n(X)$, $n = \dim X$, can never be trivial since by Hodge decomposition $b_n(X) \geq \dim_k H^0(X, \Omega_X^n) = \dim_k H^0(X, \mathcal{O}_X) = 1$. In positive characteristic, we consider the étale Betti numbers

$$b_i(X) := \dim_{\mathbb{Q}_\ell} H_{et}^i(X, \mathbb{Q}_\ell) := \varprojlim H_{et}^i(X, \mathbb{Z}/\ell^r \mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

for a prime number $\ell (\neq p)$ (see for example [27]).

Proposition 15 (cf. Prop. 8.1 [37]). *Calabi-Yau n -fold with $b_n(X) = 0$ is supersingular.*

Calabi-Yau 3-folds with trivial 3rd Betti number do exist as will be presented in section 4. We note that the converse of Prop. 15 does not hold since we have $b_2(X) = 22$ for a supersingular K3 surface X .

Recall that a variety X is called *uniruled* (or *unirational*) if there exists a dominant rational morphism $\varphi : Y \times \mathbb{P}^1 \rightarrow X$ (or $\varphi : \mathbb{P}^n \rightarrow X$) for some variety Y . In characteristic 0, a Calabi-Yau variety cannot be uniruled since otherwise ω_X cannot be trivial because of the following fact:

Theorem 16 ([28]). *Let X be a smooth projective variety over \mathbb{C} . Then X is uniruled if and only if there exists a non-empty open subset $U \subset X$ such that for all $x \in U$ there exists an irreducible curve C through x with $(K_X, C) < 0$.*

However, uniruled Calabi-Yau 3-folds do exist as will be presented in section 4.

Proposition 17 (Theorem 1.3 [16], Theorem 3.1 [17]). *A uniruled Calabi-Yau n -fold X is supersingular.*

The converse to Prop. 17 is open for $n \geq 3$. For $n = 2$, C. Liedtke showed that a supersingular K3 surface is unirational [23].

Question: Is there any relation between non-liftability to characteristic 0 and supersingularity?

3. Deformation theory of CY 3-folds in positive characteristic

3.1. Infinitesimal Lifting

Let X be a smooth projective variety over a perfect field k of $\text{char}(k) = p > 0$ and (A, \mathfrak{m}) a complete Noetherian local domain of mixed characteristic, i.e., $A = \varprojlim A/\mathfrak{m}^n$, $\text{char}(A) = 0$ and $k = A/\mathfrak{m}$. We denote the quotient field of A by K .

Example 18. A typical situation we have mostly in mind is that A is the ring of Witt vectors $(W(k), pW(k))$ over k or finite extension of $W(k)$.

3.1.1. formal spectrum $\text{Spf}(A)$

Set $A_n := A/\mathfrak{m}^{n+1}$ for $n \geq 0$. They are Artinian local rings and in particular $A_0 = k$. Also set $S_n := \text{Spec } A_n$. Then we have an increasing sequence of infinitesimal neighborhoods $S_n \subset S_{n+1}$, but all the S_n have the same underlying space, which we denote by $\text{Spf}(A)$. We define its structure sheaf as

$$\mathcal{O}_{\text{Spf}(A)} := \varprojlim \mathcal{O}_{S_n}$$

We have $\Gamma(\text{Spf}(A), \mathcal{O}_S) = A$.

3.1.2. formal lifting via infinitesimal lifting

A lift \mathcal{X} of X over A is a scheme flat over A , $\mathcal{X} \rightarrow \operatorname{Spec}(A)$, such that $X \cong \mathcal{X} \times_A k$. Its closed fiber (or special fiber) X in characteristic $\operatorname{char}(k) = p > 0$ is lifted to the generic fiber $\mathcal{X}_\eta := \mathcal{X} \times_A K$ in $\operatorname{char}(K) = 0$.

$$\begin{array}{ccccc} X & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{X}_\eta \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(k) & \hookrightarrow & \operatorname{Spec}(A) & \hookleftarrow & \operatorname{Spec}(K) \end{array}$$

One way to construct such a lifting \mathcal{X} is *infinitesimal lifting* of X . Consider the short exact sequence:

$$0 \longrightarrow \mathfrak{m}^n / \mathfrak{m}^{n+1} \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow 0 \quad (1)$$

We note that $\mathfrak{m}^n / \mathfrak{m}^{n+1} \cong \mathfrak{m} \otimes k$ is a finite k -vector space. Infinitesimal lifting is to try to lift X over $A_0 = k$ to a scheme X_1 flat over $A_1 = A / \mathfrak{m}^2$, then lift X_1 to a scheme X_2 flat over A_2 and so on to obtain a family $\{X_n\}_{n \geq 0}$ of schemes such that $X_n \otimes_{A_n} A_{n-1} = X_{n-1}$ ($n \geq 1$). Such a family $\{X_n\}$ is called a *formal family of deformations of X over A* . From this family, we obtain a formal lifting:

Proposition 19. *Given such a family $\{X_n\}_{n \geq 0}$, we obtain a Noetherian formal scheme \tilde{X} flat over $\operatorname{Spf}(A)$ such that $X_n \cong \tilde{X} \times_A A_n$ for all $0 \leq n \in \mathbb{Z}$.*

Proof. Define \tilde{X} to be the locally ringed space formed by taking the topological space X_0 together with the sheaf of rings $\mathcal{O}_{\tilde{X}} := \varprojlim \mathcal{O}_{X_n}$. See, for example, Prop. 21.1 [14] for the rest of proof. \square

The formal scheme \tilde{X} as in Prop. 19 is called a *formal lifting* of X . A formal lifting scheme is locally isomorphic to the completion of the closed fiber, but it is not always so globally. On the other hand, we say that X can be *projectively lifted* over the field K of characteristic 0 if there exists a projective scheme \mathcal{X} flat over A with $X \cong \mathcal{X} \otimes_A k$.

Then we have questions: given a projective variety X over k ,

Q1: when do we have a formal lifting \tilde{X} ?

Q2: given a formal lifting \tilde{X} , when do we have a projective lifting \mathcal{X} , whose completion along the closed fiber is \tilde{X} ?

3.1.3. obstruction to formal lifting

Definition 20 (torsor/principal homogeneous-space). Suppose that a group G acts on a non-empty set S . Then S is called a *torsor* or a *principal homogeneous space* under the action of G if there exists one (and hence all) element $s_0 \in S$ such that $G \cong S$ via $g \mapsto g(s_0)$

Now the answer to the question **Q1** is that if $H^2(X, T_X) = 0$ we always have a formal lifting of X :

Proposition 21. *The obstruction to lifting at each step lie in $H^2(X, T_X)$. If a lifting exists, the set of equivalence classes of all such is a torsor under $H^1(X, T_X)$, which means that there is a one-to-one correspondence between the set of equivalence classes of liftings and $H^1(X, T_X)$.*

Proof. Consider the short exact sequence (1) and assume that we have a scheme X_{n-1} over A_{n-1} . Then, we can show that there is just one obstruction in $H^2(X, T_X \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1})$ for the existence of a lifting X_n of X_{n-1} over A_n . Since $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finite dimensional vector space over k , we know that $H^2(X, T_X)$ implies no obstruction for the lifting. See for example Cor.10.3 [14] for the rest of the proof. \square

Here we define rigidity of scheme, which will be used later.

Definition 22 (rigid scheme). A scheme X is called *rigid* if all of its deformations over the dual numbers are trivial, i.e., $X \times_k D$, where $D := \text{Spec } k[\varepsilon]/(\varepsilon^2)$.

Proposition 23. *For a smooth scheme X over k , the following are equivalent:*

1. X is rigid;
2. all the deformations over an Artinian k -algebra A are trivial, i.e., $X \times_k \text{Spec } A$
3. $H^1(X, T_X) = 0$.

Proof. See Exercise 10.4 and Theorem 5.3. \square

3.1.4. obstructions to projective lifting

The answer to the question **Q2** is

Theorem 24. *In the situation described above, assume that $H^2(X, \mathcal{O}_X) = 0$ and $H^2(X, T_X) = 0$. Then, X can be projectively lifted to characteristic 0.*

Proof. Since $H^2(X, T_X) = 0$, we have an infinitesimal lifting \tilde{X} by Prop. 21. The obstruction to lifting an invertible sheaf lies in $H^2(X, \mathcal{O}_X)$ (see, for example, Theorem 6.4(a) [14]), which is trivial by assumption. Thus an ample line bundle L of X can be lifted to \tilde{X} . Then applying Grothendieck's algebraization theorem we obtain a projective variety \mathcal{X} whose completion along the closed fiber X is the formal lifting \tilde{X} . See Theorem. 22.1 [14] for the detail. \square

Corollary 25. *For a Calabi-Yau n -fold ($n \geq 3$), existence of a formal lifting implies existence of a projective lifting.*

Thus, projective lifting problem is a little simpler for $\dim X \geq 3$. Compare with the comments after Theorem 6.

3.2. W_2 -liftability of ordinary CY n -folds

Although there exist supersingular Calabi-Yau 3-folds which cannot be lifted over $W(k)$, or even over W_2 , the situation is a little different for ordinary Calabi-Yau varieties. Namely, F. Yobuko [46] proved recently that ordinary CY n -folds always lift over W_2 , so that in particular, for $\text{char}(k) \geq 3$, Kodaira vanishing holds by Cor. 11. Precisely,

Theorem 26 (Yobuko [46]). *Let X be a Calabi-Yau variety over an algebraically closed field k of $\text{char}(k) = p > 0$ with $ht(X) < \infty$. Then X admits a flat lift over W_2 .*

Hence by Theorem 9 we immediately obtain

Corollary 27. *Let X be an ordinary Calabi-Yau variety over an algebraically closed field k with $\dim X \leq p = \text{char}(k)$. Then Kodaira vanishing holds for X .*

The key ideas of the proof are quasi-Frobenius splitting and splitting height. To understand these notions, we need (iterated) Cartier operator, which we will explain now.

3.2.1. Cartier operator

Let X be a scheme with $p\mathcal{O}_X = 0$ with the structure morphism $\pi : X \rightarrow S$ as \mathbb{F}_p -schemes. Then we consider the absolute Frobenius morphisms F_X and F_S together with the relative Frobenius morphism $F_{X/S} : X \rightarrow X^{(p)}$:

$$\begin{array}{ccccc} X & \xleftarrow{W} & X^{(p)} & \xleftarrow{F_{X/S}} & X \\ \pi \downarrow & & \pi^{(p)} \downarrow & & \pi \downarrow \\ S & \xleftarrow{F_S} & S & \xlongequal{\quad} & S \end{array} \quad \begin{array}{l} X^{(p)} := X \times_S (S, F_S) \\ F_X = W \circ F_{X/S} \end{array}$$

Notice that since $\Omega_{X^{(p)}/S}^i = W^* \Omega_{X/S}^i$, we have $W^*(df) \in \Omega_{X^{(p)}/S}^1$ for $df \in \Omega_{X/S}^1$.

Example 28. If $S = \text{Spec } R$ and $X = \text{Spec } R[\overline{T}]$, $R[\overline{T}] := R[T_1, \dots, T_n]$, then $W : X^{(p)} \rightarrow X$ is induced by the ring homomorphism

$$W^\sharp : R[T_1, \dots, T_n] \longrightarrow R[T_1, \dots, T_n] \otimes_R (R, F_R)$$

such that $W^\sharp(T_i) = T_i$ for all i and $W^\sharp(f) = F_R(f) = f^p$ for all $f \in R$. Also, $F_{X/S}$ is induced from the ring homomorphism

$$F_{X/S}^\sharp : R[T_1, \dots, T_n] \otimes_R (R, F_R) \longrightarrow R[T_1, \dots, T_n]$$

such that $F_{X/S}^\sharp(T_i) = T_i^p$ and $F_{X/S}^\sharp(f) = f$ for $f \in R$. Then

$$\begin{aligned}
W^* \Omega_{X/S}^i &= (R[\overline{T}], F_R) \otimes_{R[\overline{T}]} \Omega_{X/S}^i \\
&= (R[\overline{T}], F_R) \otimes_{R[\overline{T}]} \left\{ \sum_{1 \leq j_1 < \dots < j_i \leq n} r_{j_1 \dots j_i} \cdot dT_{j_1} \wedge \dots \wedge dT_{j_i} \mid r_{j_1 \dots j_i} \in R[\overline{T}] \right\} \\
&= \Omega_{X^{(p)}/S}^i
\end{aligned}$$

Next we consider the de Rham complex $(\Omega_{X/S}^\bullet, d)$ and let

$$Z_1 \Omega_{X/S}^i := \text{Ker}(d : \Omega_{X/S}^i \longrightarrow \Omega_{X/S}^{i+1}) \quad \text{and} \quad B_1 \Omega_{X/S}^i := \text{Im}(d : \Omega_{X/S}^{i-1} \longrightarrow \Omega_{X/S}^i).$$

Since $d : \Omega_{X/S}^i \rightarrow \Omega_{X/S}^{i+1}$ is a $\mathcal{O}_{X^{(p)}}$ -linear map for every $i \geq 0$, $Z_1 \Omega_{X/S}^i$, $B_1 \Omega_{X/S}^i$ and the cohomology sheaf $\mathcal{H}^*(\Omega_{X/S}^\bullet)$ are all $\mathcal{O}_{X^{(p)}}$ -modules. We sometimes write as $\mathcal{H}^i((F_{X/S})_* \Omega_{X/S}^\bullet)$ to stress this fact.

Now we define the Cartier operator.

Theorem 29. *There exists a morphism*

$$C^{-1} = C_{X/S}^{-1} : \Omega_{X^{(p)}/S}^i \longrightarrow \mathcal{H}((F_{X/S})_* \Omega_{X/S}^\bullet)$$

such that

1. $C^{-1}(1) = 1$,
2. $C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$
3. $C^{-1}(W^*(df)) = \text{the cohomology class of } f^{p-1}df \text{ for } f \in \mathcal{O}_X.$

C^{-1} is isomorphic if X/S is smooth.

Proof. See, for example, Theorem. 7.1 [22]. □

Definition 30 (Cartier operator). Assume that X/S is smooth. Then the morphism $C_{X/S} : Z_1 \Omega_{X/S}^\bullet \longrightarrow \Omega_{X^{(p)}/S}^\bullet$ induced by Theorem 29

$$0 \longrightarrow B_1 \Omega_{X/S}^\bullet \longrightarrow Z_1 \Omega_{X/S}^\bullet \xrightarrow{C_{X/S}} \Omega_{X^{(p)}/S}^\bullet \longrightarrow 0 \quad (2)$$

is called the *Cartier operator*.

3.2.2. iterated Cartier operator and quasi Frobenius splitting

We recall the iterated Cartier operator defined in [20] (see also [41]). Let X be a smooth scheme over a scheme S and $n \in \mathbb{N}$. Then we define

$$\begin{aligned} X^{(p^n)} &:= X \times_S (S, F_S^n) = (X^{(p^{n-1})})^{(p)} \\ F_{X/S}^n &:= F_{X^{(p^{n-1})}/S} \circ F_{X^{(p^{n-2})}/S} \circ \cdots \circ F_{X/S} = (F_{X/S}^{n-1})^{(p)} \circ F_{X/S} \end{aligned}$$

$$\begin{array}{ccccc} X & \xleftarrow{W^n} & X^{(p^n)} & \xleftarrow{F_{X/S}^n} & X \\ \pi \downarrow & & \pi^{(p^n)} \downarrow & & \pi \downarrow \\ S & \xleftarrow{F_S^n} & S & \xlongequal{\quad} & S \end{array} \quad F_X^n = W^n \circ F_{X/S}^n$$

where the right square is expanded as follows:

$$\begin{array}{ccccccc} X^{(p^n)} & \xleftarrow{F_{X^{(p^{n-1})}/S}} & X^{(p^{n-1})} & \xleftarrow{F_{X^{(p^{n-2})}/S}} & X^{(p^{n-2})} & \longleftarrow \cdots \longleftarrow & X^{(p)} \xleftarrow{F_{X/S}} X \\ \pi^{(p^n)} \downarrow & & \pi^{(p^{n-1})} \downarrow & & \pi^{(p^{n-2})} \downarrow & & \pi^{(p)} \downarrow \\ S & \xlongequal{\quad} & S & \xlongequal{\quad} & S & \xlongequal{\quad} \cdots \xlongequal{\quad} & S \xlongequal{\quad} S \end{array}$$

Then we define

$$\begin{aligned} 0 \subset B_1 \Omega_{X/S}^i \subset \cdots \subset B_n \Omega_{X/S}^i \subset B_{n+1} \Omega_{X/S}^i \subset \cdots \\ \cdots \subset Z_{n+1} \Omega_{X/S}^i \subset Z_n \Omega_{X/S}^i \subset \cdots \subset Z_1 \Omega_{X/S}^i \subset \Omega_{X/S}^i \end{aligned}$$

as follows:

$$\begin{aligned} B_0 \Omega_{X/S}^i &= 0, \quad Z_0 \Omega_{X/S}^i = \Omega_{X/S}^i \\ B_n \Omega_{X^{(p)}/S}^i &\xrightarrow{C_{X/S}^{-1}} B_{n+1} \Omega_{X/S}^i / B_1 \Omega_{X/S}^i \\ Z_n \Omega_{X^{(p)}/S}^i &\xrightarrow{C_{X/S}^{-1}} Z_{n+1} \Omega_{X/S}^i / B_1 \Omega_{X/S}^i \end{aligned}$$

Then we have

Proposition 31 ((0.2.2.5) and (0.2.2.6.3) [20]).

$$\Omega_{X^{(p^n)}/S}^i \xrightarrow{C_{X/S}^{-n}} Z_n \Omega_{X/S}^i / B_n \Omega_{X/S}^i \quad \text{and} \quad \mathcal{H}^i(\Omega_{X^{(p^n)}/S}^\bullet) \xrightarrow{C_{X/S}^{-n}} Z_{n+1} \Omega_{X/S}^i / B_{n+1} \Omega_{X/S}^i$$

where, we write $C_{X^{(p^i)}/S}$, $i = 0, 1, 2, \dots$, simply as $C_{X/S}$.

Proof. We have, for $m = n$ or $n + 1$,

$$\begin{aligned}
Z_m \Omega_{X/S}^i / B_m \Omega_{X/S}^i &\cong \frac{Z_m \Omega_{X/S}^i / B_1 \Omega_{X/S}^i}{B_m \Omega_{X/S}^i / B_1 \Omega_{X/S}^i} \\
&\stackrel{C_{X/S}}{\cong} Z_{m-1} \Omega_{X^{(p)}/S}^i / B_{m-1} \Omega_{X^{(p)}/S}^i \cong \frac{Z_{m-1} \Omega_{X^{(p)}/S}^i / B_1 \Omega_{X^{(p)}/S}^i}{B_{m-1} \Omega_{X^{(p)}/S}^i / B_1 \Omega_{X^{(p)}/S}^i} \\
&\stackrel{C_{X/S}}{\cong} \dots \\
&\stackrel{C_{X/S}}{\cong} Z_1 \Omega_{X^{(p^{m-1})}/S}^i / B_1 \Omega_{X^{(p^{m-1})}/S}^i = \mathcal{H}^i(\Omega_{X^{(p^{m-1})}/S}^\bullet) \\
&\stackrel{C_{X/S}}{\cong} \Omega_{X^{(p^m)}/S}^i.
\end{aligned}$$

□

Remark 32. Notice that we can also define $B_{n+1} \Omega_{X/S}^i = C_{X/S}^{-1}(B_n \Omega_{X/S}^i)$ and $Z_{n+1} \Omega_{X/S}^i = \text{Ker}(dC_{X/S}^n)$ as in [41].

From now on, we set $S = \text{Spec}(k)$ and we only consider the absolute Frobenius morphism. First of all we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow (F_X)_* \mathcal{O}_X \xrightarrow{d} B_1 \Omega_X^1 \longrightarrow 0 \quad (e)_1$$

Definition 33 (Frobenius splitting [25]). X is said to be *Frobenius split* if $(e)_1$ splits.

We will now extend this notion: By pulling back along (the surjection) $B_m \Omega_X^1 \xrightarrow{C^{m-1}} B_1 \Omega_X^1$, we obtain the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & (F_X)_* \mathcal{O}_X & \xrightarrow{d} & B_1 \Omega_X^1 \longrightarrow 0 & (e)_1 \\
& & \parallel & & \uparrow & & \uparrow C^{m-1} & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{F}_m & \longrightarrow & B_m \Omega_X^1 \longrightarrow 0 & (e)_m
\end{array} \quad (3)$$

where \mathcal{F}_m is the extension of $B_m \Omega_X^1$ by \mathcal{O}_X fitting in this diagram.

Now we define

Definition 34 (splitting height and quasi Frobenius split). We define the *splitting height* $\text{sht}(X)$ of X by

$$\text{sht}(X) := \min\{m \mid (e)_m \text{ splits}\}$$

X is called *quasi Frobenius split* if $\text{sht}(X) < \infty$.

The splitting height gives a new interpretation of height for Calabi-Yau varieties.

Proposition 35 (Prop. 6 [46]). *For a Calabi-Yau variety X of $\dim X = n$, we have $ht(X) = sht(X)$*

Proof. By the following Lemma 36, $ht(X) = 1$ if and only if $sht(X) = 1$. Thus we may regard $ht(X) \geq 2$ and $sht(X) \geq 2$ and $(e)_1 \neq 0$. From the exact sequence

$$0 \longrightarrow B_{m-1}\Omega_X^1 \longrightarrow B_m\Omega_X^1 \xrightarrow{C^{m-1}} B_1\Omega_X^1 \longrightarrow 0$$

we have a long exact sequence

$$\mathrm{Ext}^1(B_1\Omega_X^1, \mathcal{O}_X) \xrightarrow{(C^{m-1})^*} \mathrm{Ext}^1(B_m\Omega_X^1, \mathcal{O}_X) \xrightarrow{\varphi_m} \mathrm{Ext}^1(B_{m-1}\Omega_X^1, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^2(B_1\Omega_X^1, \mathcal{O}_X).$$

By the definition of $(e)_m$, we have $(C^{m-1})^*((e)_1) = (e)_m$.

Now recall Prop. 3.1[41], i.e.,

$$\dim H^i(X, B_m\Omega_X^1) = \begin{cases} 0 & (i \neq n-1, n) \\ \min\{m, ht(X) - 1\} & (i = n-1, n). \end{cases}$$

and the fact that $B_m\Omega_X^1$ is locally free as a $\mathcal{O}_{X(p^m)}$ -module (see for example. Prop. 0.2.2.8(a) [20]). Then by Serre duality, we compute $\mathrm{Ext}^2(B_1\Omega_X^1, \mathcal{O}_X) = \mathrm{Ext}^2(\mathcal{O}_X, (B_1\Omega_X^1)^\vee) = H^2(X, (B_1\Omega_X^1)^\vee) = H^{n-2}(X, B_1\Omega_X^1)^\vee = 0$, which implies that φ_m is surjective. Similarly $\dim \mathrm{Ext}^1(B_m\Omega_X^1, \mathcal{O}_X) = \min\{m, ht(X) - 1\}$. Thus we know in particular that $\mathrm{Ext}^1(B_1\Omega_X^1, \mathcal{O}_X)$ is 1-dimensional and generated by $(e)_1$, which means that $\mathrm{Ker} \varphi_m = \mathrm{Im}(C^{m-1})^*$ is generated by $(e)_m$. Consequently, we know that $(e)_m = 0$ if and only if φ_m is isomorphic.

Now let $s := sht(X)$. Then $(e)_m \neq 0$ for all $m < s$. Thus if we set $d_m := \dim \mathrm{Ext}^1(B_m\Omega_X^1, \mathcal{O}_X) = \min\{m, ht(X) - 1\}$, we have that φ_m is non-isomorphic for $m < s$ and φ_s is isomorphic. Hence we have

$$d_s = d_{s-1} > d_{s-2} > \cdots > d_2 > d_1 = 1$$

and then $d_s = \min\{s, ht(X) - 1\} = \min\{s - 1, ht(X) - 1\} = d_{s-1} \geq s - 1$, from which we obtain $ht(X) = sht(X)$. \square

The following result is well-known to experts.

Lemma 36. *For a Calabi-Yau n -fold X , $ht(X) = 1$ if and only if X is Frobenius-split.*

Proof. By [41], we have

$$ht(X) = \min\{i \geq 1 \mid F : H^n(X, W_i\mathcal{O}_X) \rightarrow H^n(X, W_i\mathcal{O}_X) \text{ is non-trivial}\}$$

so that $ht(X) = 1$ is equivalent to $F : H^n(X, \mathcal{O}_X) \rightarrow H^n(X, \mathcal{O}_X)$ being non-trivial.

On the other hand, X being Frobenius-split means that

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X \xrightarrow{d} B_1\Omega_X^1 \longrightarrow 0$$

splits. Taking the dual $(-)^{\vee} = \text{Hom}(-, \mathcal{O}_X)$ it is equivalent that

$$0 \longrightarrow (B_1\Omega_X^1)^{\vee} \longrightarrow (F_*\mathcal{O}_X)^{\vee} \longrightarrow \text{Hom}(\mathcal{O}_X, \mathcal{O}_X)(\cong \mathcal{O}_X) \longrightarrow 0$$

splits, where we note that, since X is smooth, all these sheaves are locally free. This means that the identity $\text{id} : \mathcal{O}_X \rightarrow \mathcal{O}_X$ can be lifted to $F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$, so that it is equivalent that

$$H^0(X, (F_*\mathcal{O}_X)^{\vee}) \longrightarrow H^0(X, \mathcal{O}_X)$$

is non-zero. By Serre-duality and $\omega_X \cong \mathcal{O}_X$, this is equivalent with non-triviality of the Frobenius

$$H^n(X, \mathcal{O}_X) \longrightarrow H^n(X, F_*\mathcal{O}_X).$$

□

3.2.3. Mehta-Srinivas deformation theory

V.B. Mehta and V. Srinivas [26, 39] formulated a simplified version of Deligne-Illusie theory [8] of W_2 -lifting. In this subsection, we briefly summarize their results.

For any smooth variety X , the exact sequence $(e)_1$ induces the connecting homomorphism

$$\text{Ext}^1(\Omega_X^1, B_1\Omega_X^1) \xrightarrow{\delta} \text{Ext}^2(\Omega_X^1, \mathcal{O}_X). \quad (4)$$

Then there is a class

$$\text{obs}(X) \in \text{Ext}^2(\Omega_X^1, \mathcal{O}_X)$$

whose non-vanishing is the obstruction to lifting X over W_2 . Furthermore, there is a class

$$\text{obs}(X, F_X) \in \text{Ext}^1(\Omega_X^1, B_1\Omega_X^1)$$

whose non-vanishing is the obstruction to lifting the pair (X, F_X) to W_2 . This corresponds to the exact sequence (2). Now the connecting homomorphism behaves like a forgetful map:

$$\delta(\text{obs}(X, F_X)) = \text{obs}(X).$$

Then we have

Proposition 37. *If a smooth variety X is Frobenius-split, then X admits a lift over W_2 .*

Proof. If X is Frobenius-split, then $(e)_1$ splits so that $\delta = 0$. Then we have $\text{obs}(X) = \delta(\text{obs}(X, F_X)) = 0$. □

3.2.4. Proof of Theorem 26

In view of Proposition 35, we have only to show

Proposition 38 (Prop. 5 [46]). *Every quasi-Frobenius split variety admits a flat lift to W_2 .*

Proof. Let $\text{sht}(X) = m < \infty$. From the diagram (3) of the extensions $(e)_m$ and $(e)_1$, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Ext}^1(\Omega_X^1, B_1\Omega_X^1) & \xrightarrow{\delta} & \text{Ext}^2(\Omega_X^1, \mathcal{O}_X) \\ (C^{m-1})_* \uparrow & & \parallel \\ \text{Ext}^1(\Omega_X^1, B_m\Omega_X^1) & \xrightarrow{\delta_m} & \text{Ext}^2(\Omega_X^1, \mathcal{O}_X). \end{array}$$

Since $(e)_m = 0$, we have $\delta_m = 0$. Recall that $\text{obs}(X, F) \in \text{Ext}^1(\Omega_X^1, B_1\Omega_X^1)$ and if it is in the image of $(C^{m-1})_*$, the commutativity of the diagram shows that $\text{obs}(X) = \delta(\text{obs}(X, F)) = 0$ and X lifts over $W_2(k)$. But from the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_1\Omega_X^1 & \longrightarrow & Z_1\Omega_X^1 & \xrightarrow{C} & \Omega_X^1 \longrightarrow 0 \\ & & \uparrow C^{m-1} & & \uparrow C^{m-1} & & \parallel \\ 0 & \longrightarrow & B_m\Omega_X^1 & \longrightarrow & Z_m\Omega_X^1 & \xrightarrow{C^m} & \Omega_X^1 \longrightarrow 0 \end{array}$$

we immediately know $\text{obs}(X, F) \in \text{Im}((C^{m-1})_*)$ □

The following problem seems to be still open:

Question: Is an ordinary Calabi-Yau n -fold X liftable over W_n ($n \geq 3$) or even liftable to characteristic 0 ?

4. Construction of non-liftable CY 3-folds

In this section, after presenting what is known on the causes for non-liftability of Calabi-Yau varieties. After that we overview the construction techniques of the known examples of non-liftable Calabi-Yau threefolds.

4.1. what causes non-liftability?

The following two cases in which a Calabi-Yau variety is non-liftable have been known so far: one is trivial highest Betti number and the other is small resolution of liftable singular rigid Calabi-Yau variety. We will explain them in the sequel and we still have the following open question:

Question: Are these two cases equivalent?

4.1.1. non-liftability by $b_n(X) = 0$ (due to Hirokado)

By lifting to characteristic 0, we mean that $\mathcal{X} \rightarrow \operatorname{Spec}(R)$ is proper and smooth in this section.

Proposition 39. *Étale Betti numbers are preserved in lifting to characteristic 0.*

Proof. Let $\mathcal{X} \rightarrow S$ be a smooth and proper lifting of X with X_η the generic fiber. Then, by the proper and smooth base change theorem of étale cohomology, we have

$$H^i(X, \mathbb{Z}/\ell^n \mathbb{Z}) \cong H^i(X_\eta, \mathbb{Z}/\ell^n \mathbb{Z})$$

for a prime $\ell (\neq p)$. Then by taking the limit, we obtain $b_i(X) = b_i(X_\eta)$. \square

Corollary 40. *For a Calabi-Yau n -fold X in positive characteristic, $\beta_{\dim X}(X) = 0$ implies that X is non-liftable to characteristic 0.*

Proof. Let X_η be the lifting of X to characteristic 0. Since X_η is also Calabi-Yau, the Hodge decomposition and Serre duality together with $\omega_X \cong \mathcal{O}_X$ show that

$$b_n(X_\eta) = \sum_{i=0}^n \dim_K H^{n-i}(X_\eta, \Omega_{X_\eta}^i) \geq \dim_K H^n(X_\eta, \mathcal{O}_{X_\eta}) \cong H^0(X_\eta, \mathcal{O}_{X_\eta})^\vee = 1.$$

Thus $0 = b_n(X) = b_n(X_\eta) \geq 1$, a contradiction. \square

Notice that a Calabi-Yau n -fold with $b_n(X) = 0$ is supersingular by Prop. 15.

4.1.2. non-liftability by mod p reduction (due to Cynk and van Straten)

We consider the situation in the following diagram:

$$\begin{array}{ccccc} & Y & & & \\ & \pi \downarrow & & & \\ X & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_\eta \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(k) & \longrightarrow & S = \operatorname{Spec}(R) & \longleftarrow & \operatorname{Spec}(K) \end{array}$$

where X is a rigid singular Calabi-Yau 3-fold with nodes as singularity, \mathcal{X} is a lifting of X over a complete Noetherian local domain (R, \mathfrak{m}) of mixed characteristic, \mathcal{X}_η is the generic fiber, which is a (smooth) Calabi-Yau 3-fold over $K := Q(R)$, $\pi : Y \rightarrow X$ is a desingularization of nodes in X with small resolution, which is locally described as

$$X = \operatorname{Spec} k[[x, y, z, w]]/(xy - zw)$$

and

$$\pi : Y = Bl_I(X) \longrightarrow X \quad (I = (x, z) \text{ or } I = (y, z))$$

is the blow-up centered at the ideal I . Notice that the local analytic desingularization does not necessarily extend to global algebraic desingularization in the category of schemes: sometimes we have desingularization only in the category of algebraic spaces.

Then we have

Proposition 41 (Theorem 3.1 and 4.3 [5]). *Y is a non-liftable Calabi-Yau 3-fold.*

Proof. Assume that Y has a lifting \mathcal{Y} over R and set $Y_n := \mathcal{Y} \times_R R/\mathfrak{m}^{n+1}$ for $1 \leq n \in \mathbb{Z}$. Since X has at most rational singularity, we can show that there exists commutative diagrams:

$$\begin{array}{ccc} Y & \hookrightarrow & Y_n \\ \downarrow & & \downarrow \\ X & \hookrightarrow & X'_n \end{array}$$

for some X'_n (blowing-down of lifting), but since X is rigid, we must have $X'_n = X_n := \mathcal{X} \times_R R/\mathfrak{m}^{n+1}$. Since X has nodes, we have a section $\text{Spec } R/\mathfrak{m}^{n+1} \rightarrow X_n$ passing through a node. Since $\mathcal{X} = \varprojlim X_n$, \mathcal{X} also has such a section and this means that \mathcal{X} is singular, a contradiction. \square

4.2. construction (I): quotient by foliation

The first example of non-liftable Calabi-Yau 3-fold [16] has been constructed using quotient by foliation, which is a well-known method to produce purely inseparable cover.

4.2.1. foliation in positive characteristic

Most of interesting pathologies in positive characteristic are caused by purely inseparable extension of function fields. Foliation (in algebraic setting) is a tool to produce such extensions [33, 34, 9, 21].

The set of derivations $Der(K)$ has the structure of p -Lie algebra (or p -closed Lie algebra, restricted Lie algebra), which means the Lie algebra closed under Lie bracket and p th power.

Definition 42 (exponent and p -basis). Let K/k be an extension of fields of characteristic $p > 0$.

- (i) We say that $a \in K$ is of *exponent* $e(\geq 0)$ if a is purely inseparable over k and its minimal polynomial is in the form of $x^{p^e} - c$ for some $c \in k - \{0\}$. K/k is called purely inseparable of *exponent* $\leq e$ if every element $a \in K$ is of exponent $\leq e$.
- (ii) a set $B = \{\rho_1, \dots, \rho_m\} \subset K$ is a *p -basis* of the extension if $K = k(K^p)(B)$, equivalently $\{\rho_1^{k_1} \cdots \rho_m^{k_m} \mid 0 \leq k_i < p\}$ is the $k(K^p)$ basis of degree p^m extension $K/k(K^p)$.

For an extension K/k of fields in characteristic $p > 0$, we have $D : K/k(K^p) \rightarrow K/k(K^p)$ for a derivation $D : K/k \rightarrow K/k$ over k . Namely, we can consider D to be a derivation over $k(K^p)$ and $K/k(K^p)$ is of exponent ≤ 1 since the extension $K/k(K^p)$ is described by p -basis. Thus when we consider relation between derivation and purely inseparable extension, we have only to consider the action of a derivation on p -basis of extension of exponent ≤ 1 .

Now let K be a field of $\text{char}(K) = p > 0$. According to Jacobson's Galois theory for purely inseparable extensions (of exponent one) [21], there is a one-to-one correspondence between

$$\mathcal{E} := \left\{ \begin{array}{l} L \subset K \\ \text{(subfield)} \end{array} \mid \begin{array}{l} K/L \text{ is purely inseparable of exponent } \leq 1, \\ (K : L) < \infty \end{array} \right\}$$

and

$$\mathcal{D} := \{ \mathcal{L} : p\text{-Lie algebra of derivations over } K \mid \dim_K \mathcal{L} < \infty \}$$

Namely,

$$\begin{array}{ccc} \mathcal{D} & \xleftrightarrow{1:1} & \mathcal{E} \\ \mathcal{L} & \longrightarrow & K^{\mathcal{L}} = \{a \in K \mid D(a) = 0 \ (\forall D \in \mathcal{L})\} \\ \mathcal{L}_L = \{D : K/L \rightarrow K/L : \text{derivation}\} & \longleftarrow & L \end{array}$$

Jacobson's theory has been reformulated to foliation of algebraic varieties by T. Ekedahl [9].

Definition 43 (foliation in characteristic p). Let X be a normal variety over an algebraically closed field k of $\text{char}(k) = p > 0$. Then a coherent subsheaf $\mathcal{F} \subset T_X$ is called a *foliation* if it is closed under Lie bracket and p th power

Definition 44 (quotient by a foliation). Let $\mathcal{F} \subset T_X$ be a foliation. Set

$$\text{Ann}(\mathcal{F})(U) := \{a \in \mathcal{O}_X(U) \mid D(a) = 0 \ (\forall D \in \mathcal{F}(U))\} \quad (\forall U \subset X : \text{open})$$

Then we call $Y := \text{Spec } \text{Ann}(\mathcal{F})$ the *quotient* of X by the foliation \mathcal{F} .

Proposition 45. *Under the above hypothesis, we have*

- (i) *if $\text{rk } \mathcal{F} = r$, then $[k(X) : k(Y)] = p^r$,*
- (ii) *if X is smooth, Y is also smooth if and only if \mathcal{F} is a subbundle, i.e., T_X/\mathcal{F} is locally free.*

Definition 46 ($\text{Sing}(\mathcal{F})$). Based on Prop. 45(ii), we consider the maximal Zariski open subset $U \subset X$ such that \mathcal{F} is a subbundle of $T_X|_U$. Then set $\text{Sing}(\mathcal{F}) = X - U$, which we call the *singular locus* of the foliation \mathcal{F} .

$\text{Sing}(\mathcal{F})$ is the points of X that will be the singular locus of the quotient $\text{Spec}(\text{Ann}(\mathcal{F}))$.

4.2.2. construction of Hirokado 99 variety

We show an outline of the construction of a non-liftable Calabi-Yau 3-fold in characteristic 3 [16]. This is an application of quotient by foliation.

Step 1 Consider the derivation

$$\delta := (x^p - x) \frac{\partial}{\partial x} + (y^p - y) \frac{\partial}{\partial y} + (z^p - z) \frac{\partial}{\partial z}.$$

on the affine open subset $\mathbb{A}^3 := \text{Spec } k[x, y, z] \subset \mathbb{P}^3$, which can be naturally extended to \mathbb{P}^3 . We know that $\delta^p = (-1)^{p-1} \delta$, so that we obtain a rank one foliation $\mathcal{F} := \langle \delta \rangle (\subset T_{\mathbb{P}^3})$. We know that $\text{Sing}(\mathcal{F}) = \mathbb{P}_{\mathbb{F}_p}^3 (\subset \mathbb{P}^3)$, which are $p^3 + p^2 + p + 1$ isolated points.

Step 2 Each singular point in $\text{Sing}(\mathcal{F})$ can be resolved by one point blow-up. Then we have the following diagram

$$\begin{array}{ccccc} S & \xrightarrow{g} & X & \xrightarrow{\bar{g}} & S^{(p)} \\ \pi \downarrow & & \bar{\pi} \downarrow & & \downarrow \\ \mathbb{P}^3 & \xrightarrow{g_0} & V & \xrightarrow{\bar{g}_0} & (\mathbb{P}^3)^{(p)} \end{array}$$

where (1) $\pi : S \rightarrow \mathbb{P}^3$ is the blow-ups at $p^3 + p^2 + p + 1$ singular points in $\text{Sing}(\mathcal{F})$, (2) g is the quotient morphism (of degree p) induced by the foliation $\pi^* \mathcal{F}$, which has no singular point, (3) g_0 is the quotient morphism (of degree p) induced by \mathcal{F} , (4) $\bar{\pi}$ is the naturally induced birational morphism, which is a desingularization of V , and (5) \bar{g} and \bar{g}_0 are the morphisms of degree p^2 in the relative Frobenius morphisms $\bar{g} \circ g$ and $\bar{g}_0 \circ g_0$.

Step 3 We have

$$g^* \omega_X \cong \pi^* \mathcal{O}_{\mathbb{P}^3}((p-1)^2 - 4) \otimes \mathcal{O}_S(D)$$

where $D = (3-p) \sum_{i=1}^{p^3+p^2+p+1} E_i$ (E_i : exceptional divisor of π).

In particular, for $p = 3$, we have $g^* \omega_X \cong \pi^* \mathcal{O}_{\mathbb{P}^3}$ from which we obtain $\omega_X \cong \mathcal{O}_X$ together with $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Thus X is a Calabi-Yau 3-fold.

This example has the following properties: (1) $b_3(X) = 0$ so that non-liftable and supersingular, (2) X is unirational and hence simply connected, (3) Hodge symmetry holds and moreover (4) weak H^1 -Kodaira vanishing holds, i.e., $H^1(X, L^{-1}) = 0$ for ample L such that $H^0(X, L^p) \neq 0$ [43], however, according to [10], it is not liftable to W_2 .

4.3. construction (II): supersingular K3 pencil over \mathbb{P}^1

Let k be an algebraically closed field of characteristic $p > 0$. Let $f : X \rightarrow \mathbb{P}_k^1$ be a smooth morphism such that $\omega_X = \mathcal{O}_X$ and all geometric fibers $X_{\bar{t}}$ ($t \in \mathbb{P}_k^1$) are K3 surfaces with Picard number $\rho(X_{\bar{t}}) = 22$, i.e., supersingular K3 surfaces. Such a variety X is necessarily a projective Calabi-Yau 3-fold (see Prop. 1.1, Prop. 1.7 [37]).

Moreover, we have $b_3(X) = 0$ so that non-liftable to characteristic 0, X is unirational and hence simply connected and weak H^1 -Kodaira vanishing holds [44], however, according to [10], it is not liftable to W_2 .

Such a 3-fold X is called a *Schröer variety*.

4.3.1. construction of Schröer varieties

The outline of the Schröer's construction is as follows. The idea is taking quotient of (a modified version of) Moret-Bailly construction of relative abelian surfaces over \mathbb{P}^1 [29] by involution to obtain Kummer surface pencil if $p = 3$. For $p = 2$, we consider a generalized Kummer surface instead of ordinary Kummer surface.

Step 1 Let $A = E_1 \times E_2$, where E_i are supersingular elliptic curves. Then we have a subgroup scheme

$$K(L) := \alpha_p \times \alpha_p \subset A \quad \text{where } \alpha_p := \text{Spec } k[X]/(X^p)$$

Then we take the product with \mathbb{P}^1 :

$$K(L) \times \mathbb{P}^1 = \text{Spec } \mathcal{O}_{\mathbb{P}^1}[x, y]/(x^p, y^p) \subset A \times \mathbb{P}^1.$$

Step 2 We take

$$H = \text{Spec } \mathcal{O}_{\mathbb{P}^1}[x, y]/(x^p, y^p, tx - sy) \subset K(L) \times \mathbb{P}^1$$

where $[s : t]$ is the projective coordinates of \mathbb{P}^1 . Actually, H is a group scheme of height one corresponding to p -Lie subalgebras $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \subset \text{Lie}(A \times \mathbb{P}^1)$. Now take $\tilde{X} := (A \times \mathbb{P}^1)/H$, which is a relative abelian surface on \mathbb{P}^1 .

Step 3 Take the fiberwise quotient by

- the involution induced from $(x, y) \mapsto (-x, -y)$ on A together with desingularization (if $p = 3$), or
- an automorphism $\sigma : S \rightarrow S$ of order 3 together with desingularization (if $p = 2$)

we obtain the (generalized) Kummer surface pencil $\pi : X \rightarrow \mathbb{P}^1$, which is a Calabi-Yau threefold.

4.4. construction (III): Schoen type examples

This construction is originated from [35] and applied to produce non-liftable Calabi-Yau 3-folds by many authors including Schoen himself [36]. Basic ideal is as follows:

Step 1 Take the fiber product

$$W := Y_1 \times_{\mathbb{P}^1} Y_2$$

where $\pi_i : Y_i \rightarrow \mathbb{P}^1$ are (rational quasi-)elliptic surfaces with section (to use Weierstrass form).

Step 2 Carry out desingularization of W : Singular points of W come from singular fibers of Y_i . In some cases, the singularities of W can be resolved with small resolution.

Step 3 The desingularization \overline{W} of W is a Calabi-Yau 3-fold with desired properties.

4.4.1. examples by Hirokado-Ito-Saito

In the fiber product $Y_1 \times_{\mathbb{P}^1} Y_2$, $\pi_i : Y_i \rightarrow \mathbb{P}^1$ are (1) a rational quasi-elliptic surface with section Y_1 and a rational elliptic surface with section Y_2 in characteristic $p = 2, 3$ in [17] or (2) quasi-elliptic rational surfaces Y_i with section in characteristic $p = 3$ [18] and $p = 2$ [19].

For these examples, we have $b_3(X) = 0$ (hence non-liftable), and have structures of fibration over \mathbb{P}^1 or \mathbb{P}^2 . Also they are unirational, hence simply connected.

4.4.2. examples by Cynk, van Straten and Schütt

We can apply the theory presented in section 4.1.2 to Schoen type construction. Then without using the condition $b_3(X) = 0$ we can assure non-liftability. But $b_3(X) = 0$ for some of the examples are proved in [36]. Also, since we use small resolution, most of the obtained examples are not projective but algebraic spaces.

4.5. construction (IV): double cover of \mathbb{P}^3 – examples by Cynk and van Straten

Based on the theory presented in section 4.1.2, we can construct the examples in the following way (at least conceptually):

Step 1 (CY3-fold over $\text{char} = 0$) Let X_0 be a Calabi-Yau 3-fold X_0 over a field K of characteristic 0.

Step 2 (find a good reduction– preparation) Let $\mathcal{X} \rightarrow \text{Spec}(R)$ be a scheme over over a suitable \mathbb{Z} -algebra R with $K = Q(R)$ such that X_0 is its generic fiber: $\mathcal{X}_\eta = X_0$.

Step 3 (find a CY reduction) There exists a non-empty Zariski open set $V \subset \text{Spec}(R)$ such that the special fiber $X_P := \mathcal{X} \times_R k(P)$ is non-singular Calabi-Yau for $P \in V$.

Step 4 (find a rigid singular fiber X_P) Now we find $P \in \operatorname{Spec}(R) - V$ such that X_P is Calabi-Yau with node as singularity.

Step 5 (small resolution) Now take a desingularization $\varphi : Y \rightarrow X_P$ with small resolution. Then Y is the desired example.

In fact, we first construct \mathcal{X} and then find X_0 . \mathcal{X} is obtained by double cover of \mathbb{P}^3 ramified at some particular octic or Clebsch diagonal cubic.

The examples in 4.4.2 are also constructed with this method.

4.6. Raynaud-Mukai construction cannot produce CY 3-folds

The famous counter-example to Kodaira vanishing given by Raynaud [32] has been generalized by Mukai [30] and we can produce 3-folds which is non-liftable even over W_2 (see Corollary 11). Then it was hoped that non-liftable Calabi-Yau 3-folds on which Kodaira vanishing does not hold could be produced with this method. However, it turned out that it is impossible [42, 44].

Whether there exists a non-liftable Calabi-Yau variety on which Kodaira vanishing does not hold seems to be still open.

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